A prototype for dS/CFT

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Abstract

We consider dS_2/CFT_1 where the asymptotic symmetry group of the de Sitter spacetime contains the Virasoro algebra. We construct representations of the Virasoro algebra realized in the Fock space of a massive scalar field in de Sitter, built as excitations of the Euclidean vacuum state. These representations are unitary, without highest weight, and have vanishing central charge. They provide a prototype for a new class of conformal field theories dual to de Sitter backgrounds in string theory. The mapping of operators in the CFT to bulk quantities is described in detail. We comment on the extension to dS_3/CFT_2.

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I.   INTRODUCTION

The main tool string theorists have at their disposal in understanding nonperturbative spacetime physics is duality with large N quantum field theories, such as AdS/CFT and Matrix theory. The generalization of these methods to de Sitter space [1] confronts us with various interesting challenges. In particular, the modes of a generic massive scalar field in d-dimensional de Sitter transform under the isometry group $SO(d, 1)$ as principal series representations [2, 3], so the corresponding operators in the dual CFT must transform in the same way under the global conformal group [4, 5, 6, 7]. Such representations do not appear in the conformal field theories common in string theory and statistical mechanics. It is therefore an important open problem to better understand this new class of conformal field theories.

In this paper, we construct perhaps the simplest example of a free conformal field theory that gives rise to principal series representations. We begin with two-dimensional de Sitter where the $SL(2, \mathbb{R})$ isometry group appears as a subgroup of a much larger asymptotic symmetry algebra, the Virasoro algebra. The representations of the full Virasoro algebra are realized on the multiparticle Fock space of a free massive scalar field in de Sitter [28], and are consequently unitary. The Virasoro algebra is found to close with vanishing central term. The representations are presented in a basis obtained in a spherical slicing of de Sitter, and in another basis obtained using a flat slicing.

We then consider the analogous computation for the case of three-dimensional de Sitter with a flat slicing, where the $SL(2, \mathbb{C})$ isometry group lifts to an asymptotic symmetry group containing commuting left/right copies of the Virasoro algebra. The Virasoro currents are well-defined locally, but we find only the global $SL(2, \mathbb{C})$ subgroup gives rise to properly-defined charges. The bulk matter modes therefore do not yield representations of Virasoro. The local Virasoro symmetry nevertheless constrains the properties of CFT correlators, as in standard CFT.

We briefly consider the problem of counting the horizon entropy in de Sitter using this framework. On a positive note, the one-to-one mapping between CFT operators and scalar particle states allows one to explicitly map the thermal entropy of a static patch observer into a CFT computation. Other authors have speculated that applying the Cardy formula to this counting problem will yield the correct, finite horizon entropy of de Sitter [8, 9, 10, 11].

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However since the mapping between the CFT counting problem and the bulk scalar field counting problem really is one-to-one, the result is divergent. We speculate on further physical input needed to regulate this result.

II. BASIC SETUP

Let us collect together a few elementary results on mode expansions of a scalar field in d-dimensional de Sitter space (further details may be found in [10]). Throughout the paper we will take the mass of the field to lie in the generic range $m > (d-1)/2$, which is the range associated with principal unitary series representations. Moreover, we will assume all expansions are made around the Bunch-Davies/Euclidean vacuum state.

A. Spherical slicing

The spacetime metric is

$$ds^2 = -d\tau^2 + \cosh^2\tau \; d\Omega_{d-1}^2 \; ,$$

with $d\Omega_{d-1}^2$ the standard metric on the $d-1$ sphere. These coordinates cover the complete de Sitter spacetime.

A massive scalar field has a mode expansion in terms of creation/annihilation operators a_{lj}

$$\phi = \sum_{l,j} y_l(\tau) Y_{lj}(\Omega) a_{lj} + \text{h.c.} \; , \tag{1}$$

where the $Y_{lj}(\Omega)$ are spherical harmonics, labeled by a non-negative integer l and an index $j = (j_1, \cdots, j_{d-2})$. The a_{lj} annihilate the Euclidean vacuum. The y_l's are solutions to a hypergeometric equation. We will only need their expansion in the limit $\tau \to -\infty$

$$y_l(\tau) \sim A_l e^{h_-\tau} + B_l e^{h_+\tau}, \tag{2}$$

with

$$h_\pm = \frac{d-1}{2} \pm i\mu, \qquad \mu = \sqrt{m^2 - \frac{(d-1)^2}{4}} \; , \tag{3}$$

and A_l and B_l constants

$$A_l = \frac{2^{\frac{d-2}{2}} e^{i\theta_l}}{\sqrt{\mu(1 - e^{-2\pi\mu})}} \; , \qquad B_l = \frac{2^{\frac{d-2}{2}} e^{-i\theta_l + i\pi(L + \frac{d-1}{2})}}{\sqrt{\mu(e^{2\pi\mu} - 1)}} \; ,$$

defined in terms of the phase

$$e^{-2i\theta_l} = e^{i\pi(l - \frac{d-1}{2})} \frac{\Gamma(-i\mu)\Gamma(l + \frac{d-1}{2} + i\mu)}{\Gamma(i\mu)\Gamma(l + \frac{d-1}{2} - i\mu)} \ .$$

The normalization is chosen so that the modes are orthonormal with respect to the Klein-Gordon inner product

$$(\psi, \phi) = -i \int d\Sigma^\mu (\psi \overleftrightarrow{\partial_\mu} \phi^*) \ . \tag{4}$$

B. Flat slicing

The spacetime metric for d-dimensional de Sitter is

$$ds^2 = -dt^2 + e^{-2t} dx_{d-1}^2 \ , \tag{5}$$

where dx_{d-1}^2 is the standard metric on flat $(d-1)$-dimensional Euclidean space. The coordinates only cover the lower triangle of de Sitter space, the time-reverse of the inflating patch.

Unlike the case of global coordinates spatial slices are now non-compact, so it is most convenient to make mode functions plane-wave normalizable. We therefore choose a basis for the spatial mode functions $e^{ik\cdot x}$ where $k \in \mathbb{R}^{d-1}$ so that

$$\phi(x, t) = \int \frac{dk^{d-1}}{(2\pi)^{d-1}} \, a_k f_k(t) e^{ik\cdot x} + a_k^\dagger f_k^*(t) e^{-ik\cdot x} \ , \tag{6}$$

where the time-dependent mode functions are now [12]

$$f_k(t) = \frac{\sqrt{\pi}}{2} e^{\frac{\pi\mu}{2}} e^{\frac{d-1}{2}t} H_{i\mu}^{(2)}(|k|e^t) \ ,$$

with $H_{i\mu}^{(2)}$ a Hankel function of the second kind. The modes $f_k(t)e^{ik\cdot x}$ have a plane-wave normalization with respect to the Klein-Gordon inner product (4)

$$(k, k') = (2\pi)^{d-1} \delta^{d-1}(k - k') \ .$$

In the limit $t \to -\infty$

$$\begin{aligned} f_k(t) &\sim A_k e^{h_- t} + B_k e^{h_+ t} \\ A_k &= \frac{i\Gamma(i\mu)}{2\sqrt{\pi}} e^{\frac{\pi\mu}{2}} \left(\frac{|k|}{2}\right)^{-i\mu} , \qquad B_k = \frac{i\Gamma(-i\mu)}{2\sqrt{\pi}} e^{-\frac{\pi\mu}{2}} \left(\frac{|k|}{2}\right)^{i\mu} . \end{aligned} \tag{7}$$

Note the much simpler k dependence versus (2). The following identity is useful in the manipulation of these mode functions:

$$\Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin \pi z} \ .$$

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III. TWO DIMENSIONS

A.   Asymptotic symmetry group

We will first work with two-dimensional de Sitter in global coordinates,

$$ds^2 = -d\tau^2 + \cosh^2 \tau \ d\theta^2 \ ,$$

with θ periodically identified up to 2π. Consider the group of diffeomorphisms that leave fixed the asymptotic conditions

$$
\begin{aligned}
g_{\tau\tau} &= -1 + O\left(e^{2\tau}\right) \\
g_{\tau\theta} &= O\left(1\right) \\
g_{\theta\theta} &= \frac{1}{2}e^{-2\tau} + O\left(1\right)
\end{aligned}
\tag{8}
$$

at past infinity $\tau \to -\infty$. This group is generated by the vector fields whose asymptotic behavior is

$$L_U = U(\theta)\frac{\partial}{\partial\theta} + U'(\theta)\frac{\partial}{\partial\tau} + O\left(e^{2\tau}\right), \tag{9}$$

where $U(\theta)$ is an arbitrary function on the circle. For the choice $U(\theta) = ie^{in\theta}$, the L_n's generate a centerless Virasoro (or Witt) algebra

$$[L_m, L_n] = (m - n)L_{m+n} \ .$$

$L_0, L_{\pm 1}$ are the vector fields associated with the dS_2 isometry group $SL(2,\mathbb{R})$. Note that this choice of global generators is related by a nontrivial isomorphism to the choice studied in [4, 5, 10]. In comparing formulas in the two notations, bear in mind that

$$
\begin{aligned}
\tilde{L}_0 &= \frac{1}{2}\left(L_{-1} - L_1\right) \\
\tilde{L}_1 &= -\frac{i}{2}\left(L_1 + L_{-1} + 2L_0\right) \\
\tilde{L}_{-1} &= \frac{i}{2}\left(L_1 + L_{-1} - 2L_0\right) \ ,
\end{aligned}
$$

where the $\tilde{L}_n$ are the Virasoro generators of [4, 5, 10]. Therefore the generator of time translations in the static patch will be $(L_{-1} - L_1)/2$ in the present notation. Furthermore, in the present notation, Hermitian conjugation will act as $L_n \to L_{-n}$, as in conventional CFT.

The action of the asymptotic symmetry group on the gravitational degrees of freedom is generated by a boundary stress energy tensor studied by Brown and York [13]. A version of this for two-dimensional de Sitter, realized as a solution of dilaton gravity, has been studied in [14, 15]. The algebra generated by moments of this stress tensor is Virasoro, with a nontrivial central charge.

B. Representations of Virasoro algebra

It is helpful to first briefly review the AdS/CFT case. In the simplest situation, an operator transforming as a conformal primary with positive real conformal weight is mapped to a single particle mode in the bulk of AdS. The isometries of AdS are identified with elements of the global conformal group on the boundary, and the particle modes form a representation of this group. For AdS_2 this isometry group is enhanced to a larger asymptotic symmetry group which is the full Virasoro algebra, and L_0 generates light-cone time translations [16]. Furthermore, descendant operators correspond to multi-particle states with larger values of L_0. Thus a highest weight Virasoro module maps into multiparticle states in the bulk, which are descendants of the lightest state [17].

Let us follow the same strategy in de Sitter. We begin with a scalar field excitation in the bulk, and fix the scalar mass m. We will then proceed to act on this state with the Virasoro generators, and deduce the structure of the full Virasoro module. An important difference with AdS is immediately apparent: $L_0 = i\frac{\partial}{\partial\theta}$ generates translations around the spacelike circle, so the eigenvalues of L_0 will be unbounded above and below.

The scalar field stress energy tensor is

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}g^{\lambda\rho}\partial_\rho\phi\partial_\lambda\phi - \frac{1}{2}g_{\mu\nu}m^2\phi^2 \ ,$$

so the matter contribution to the Virasoro generators (9) takes the form

$$L_U = -\frac{i}{2}e^{-\tau}\int_0^{2\pi} d\theta \ \left(U(\theta)\phi'\dot{\phi} + \frac{1}{2}U'\left(\dot{\phi}^2 + m^2\phi^2\right)\right) \ , \qquad (10)$$

where we have dropped terms subleading as $\tau \to -\infty$.

To make this expression well-defined, we must specify a normal-ordering prescription. We define normal ordering with respect to the Bunch-Davies, or Euclidean, vacuum state. The

mode decomposition is (1) – and we will set $l = |k|$, where $k \in \mathbb{Z}$ is the quantum number of momentum around the spatial circle.

We normal order with respect to this vacuum state, i.e., we order a_k to the right of $a_k^\dagger$'s when defining the generators (10), and we denote the normal ordered generators $: L_U :$. It is straightforward to verify the $: L_U :$'s generate an algebra of Virasoro type. The only potential subtlety is computing the central charge. This is discussed in more detail in appendix A, where it is shown the central charge vanishes.

Time-dependent coefficients seem to appear when the generators (10) are written in terms of creation and annihilation operators, using the mode expansion (1) and the asymptotic form (2). However this time dependence actually completely cancels out of the generators, and the resulting coefficients are recorded in appendix B. The generic $: L_n :$ involves terms of the three basic types $a^\dagger a$, $a^\dagger a^\dagger$ and aa, and consequently mixes states with different particle number (notice that states with an even/odd number of particles transform irreducibly). For the global ($n = 0, \pm 1$) generators, however, the terms of the second and third type can be seen to cancel out, implying that single-particle states by themselves carry a representation of the isometry group, in consonance with the results of [2, 3, 4, 5, 6, 7].

One might wonder if the different α-vacua of field theory in de Sitter give rise to different representations. As discussed in [4] at the level of the global conformal group $SL(2, \mathbb{R})$, the Euclidean vacuum is picked out by demanding the bulk-to-boundary mapping be local in the boundary coordinates. This jibes well with the expected instabilities of interacting α-vacua once coupling to bulk gravity is included [18].

The bulk-to-boundary map was described in [4] for the case of single-particle states. There the dS_2/CFT_1 mapping is performed by noticing there are two equivalent representations expressed in terms of functions of θ, τ decomposed in modes $f_k(\tau)e^{ik\theta}$ or just in terms of "boundary" modes $e^{ik\theta}$. The factors $f_k(\tau)$ can then be viewed as the bulk-to-boundary propagator in momentum space. In the present context, time-independence of the conformal generators when written in terms of creation and annihilation operators a_k, $a_k^\dagger$ is important for the consistency of these dual pictures.

The new feature with the full Virasoro representations is that states with different particle number are part of the same representation. Nevertheless, once the generators are written as time-independent linear combinations quadratic in creation/annihilation operators, the same basic picture of the bulk-to-boundary mapping will carry over where $f_k(\tau)$ factors play the

Bulk dS	Boundary CFT
$\{L_n\}\,|E\rangle$	1+descendants
$\{L_n\}\,f_k^*(\tau)a_k^\dagger|E\rangle$	$\mathcal{O}_k$+descendants
$\{L_n\}\,f_k^*(\tau)f_{k'}^*(\tau)a_k^\dagger a_{k'}^\dagger|E\rangle$	$\mathcal{O}_{k,k'}$+descendants

Table I: The de Sitter state/CFT operator mapping. Here $\{L_n\}$ denotes an element in the universal enveloping algebra of Virasoro. The CFT operator $\mathcal{O}_k$ denotes a component of a principal series representation of $SL(2,\mathbb{R})$ with spatial momentum k. The operator $\mathcal{O}_{k,k'}$ denotes a component in the tensor product of two principal series representations of $SL(2,\mathbb{R})$ labelled by spatial momenta k and k'. A similar pattern continues for higher tensor products.

role of bulk-to-boundary propagators. We reiterate we are only treating a free scalar field, so interactions are ignored. Generically we will get distinct representations corresponding to Virasoro acting on distinct multiparticle excitations of the Euclidean vacuum as summarized in table I.

C. Unitarity

Ordinarily, one concludes that unitary representations of Virasoro with vanishing central charge are trivial. However the representation discussed here is not of highest weight type, so let us re-examine this conclusion.

The symmetry algebra changes particle number, so in particular the vacuum is not invariant. The representation space lives in a subspace of the Fock space of free particles in the Euclidean vacuum. The natural norm on this entire Fock space is defined in momentum space as

$$\prod_{k=-\infty}^{\infty} \langle n_k'|n_k\rangle = \prod_{k=-\infty}^{\infty} \delta_{n_k,n_k'} \;, \tag{11}$$

where n_k is the occupation number of mode k and

$$|n_k\rangle = \frac{1}{\sqrt{n_k!}}a_k^{\dagger n_k}|E\rangle$$

with $|E\rangle$ the Euclidean vacuum. The notion of Hermitian conjugation induced by this norm on the Virasoro generators is $L_n^\dagger = L_{-n}$. The states $L_n|\psi\rangle$ therefore have positive norm, for

$|\psi\rangle$ a state with finite particle number. In particular

$$\langle E|L_{-n}L_n|E\rangle = 2n\langle E|L_0|E\rangle + \langle E|L_nL_{-n}|E\rangle \ ,$$

and while the first term on the right hand side vanishes, the second term is non-vanishing. For a highest weight representation, on the other hand, both terms would vanish. In that case, the representation that includes the vacuum is nontrivial only for nonzero central charge.

D. Planar coordinates

The above discussion of Virasoro generators can be generalized to planar coordinates

$$ds^2 = -dt^2 + e^{-2t}dz^2$$

where $z \in \mathbb{R}$. The main difference is finding the most convenient choice of mode functions, and choosing a basis for the Virasoro generators. Unlike the case of global coordinates spatial slices are now non-compact, so it is most convenient to make mode functions plane-wave normalizable. We therefore choose a basis for the spatial mode functions e^{ikz} where $k \in \mathbb{R}$ as in (6).

To keep the action of the Virasoro generators on these mode functions as simple as possible, we likewise choose a basis $U(z) = ie^{ipz}$ where $p \in \mathbb{R}$. The main new feature is that Virasoro generators are labelled by a real parameter p rather than the familiar integer parameter. Otherwise the calculations of the previous subsections go through with sums replaced by integrals and only minor modifications. Explicit expressions for the Virasoro generators are given in appendix B. Note that if we instead choose the usual basis of Virasoro generators $U = -z^n$ the action of the generators will take us outside the space of plane-wave normalizable functions.

IV. THREE DIMENSIONS

A. Asymptotic symmetry group

For the case of three-dimensional de Sitter, it is more convenient to use flat spatial sections, to exhibit the left-right decomposition of the CFT_2. Introducing $z = x_1 + ix_2$

versus (5), the metric is

$$ds^2 = -dt^2 + e^{-2t}dz\, d\bar{z} \ .$$

We consider the asymptotic symmetry group leaving fixed the conditions [1]

$$
\begin{aligned}
g_{z\bar{z}} &= \frac{1}{2}e^{-2t} + O(1) \\
g_{tt} &= -1 + O(e^{2t}) \\
g_{zz} &= O(1) \\
g_{tz} &= O(1)
\end{aligned}
$$

as $t \to -\infty$. This symmetry group is the group of diffeomorphisms generated by the vector fields

$$\zeta_U = U\partial_z + \frac{1}{2}\partial_z U \partial_t + O(e^{2t}) + \text{complex conjugate} \ ,$$

where U is holomorphic. The basis choice $U(z) = -z^{n+1}$ yields the standard mutually commuting sets of Virasoro generators $L_n, \bar{L}_n$.

B. Representations of Virasoro left+right?

By analogy with the case of dS_2 analyzed in the previous section, we might expect the multiparticle states associated with a bulk scalar field to belong to a representation of left+right Virasoro that decomposes into principal series representations of the global isometry group $SL(2,\mathbb{C})$. The Virasoro generators now take the form

$$L_U = -i\ e^{-2t} \int d^2z \left(U\partial_z\phi\partial_t\phi + \frac{1}{4}\partial_z U \left((\partial_t\phi)^2 + m^2\phi^2\right) \right) \tag{12}$$

(and likewise for $\bar{L}_U$), again dropping terms that vanish as $t \to -\infty$. The analog of the calculation described in appendix A implies these operators generate a Virasoro algebra with vanishing central charge. These operators have a well-defined action on local operators in the CFT.

However, since the function $U(z)$ necessarily has non-simple poles for all but the global $SL(2,\mathbb{C})$ transformations, the spatial integral will diverge for states with smooth distributions of matter, such as we have for a one-particle excitation of the Euclidean vacuum. (The divergence is of course also visible in momentum space, if we attempt to expand the generators in terms of creation and annihilation operators as in appendix B.) So, even though

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the L_n operators (12) satisfy the expected commutation relations, their matrix elements between generic multiparticle states are ill-defined unless $n = 0, \pm 1$.

The key issue here is the choice of boundary conditions. Recall that the Brown-York tensor τ_{ij} (with i, j running over the spatial directions) [13] satisfies

$$D_i \tau^{ij} = -T^{0j} \ ,$$

where $T^{\mu\nu}$ is the matter stress energy tensor. This equation may be integrated across a two-dimensional region at past infinity to give

$$H_{C_1}(\zeta) - H_{C_2}(\zeta) = -e^{-2t} \int_{\Sigma} d^2z \ T^{0i} \zeta_j \ , \tag{13}$$

with

$$H_C(\zeta) = \frac{1}{2\pi i} \oint_C dz \ \tau_{zz} \zeta^z \ .$$

If we assume matter stress energy is negligible at past infinity, the right hand side of (13) vanishes, and the H's are contour-independent [29]. They generate an algebra that is a direct sum of holomorphic and anti-holomorphic copies of the Virasoro algebra with a non-trivial central charge [1, 19].

However, if we want to describe even a one-particle state, then the matter stress-energy is not completely negligible at past infinity, and so we cannot drop the terms on the right-hand side of (13). As we have already noted, for all but the global $SL(2, \mathbb{C})$ transformations the integral over Σ will be ill-defined. We conclude therefore that if we allow for such boundary conditions on the matter fields, then the asymptotic symmetry group collapses to $SL(2, \mathbb{C})$ and there is no sensible extension of the principal series representations to the algebra of left/right Virasoro.

Note that the crucial difference with the two-dimensional case analyzed in the previous section is that the dS_2 asymptotic symmetry group is in fact the full group of diffeomorphisms on the circle, for which one can choose a normalizable basis and obtain well-defined integrals.

V. DISCUSSION

Since we have described the mapping from representations of Virasoro to particle states in de Sitter, as summarized in table I, we can identify in the conformal field theory the

procedure leading to entropy counting. States in the asymptotic past will form representations of the isometry group and can be propagated to arbitrary times using the isometry generators.

For definiteness, let us restrict attention to the case of dS_2 in spherical coordinates, though generalization to the flat slicing is straightforward. One may choose a basis of these states which diagonalizes the generator of time translations in the static patch, $(L_{-1} - L_1)/2$. This change of basis is accomplished as follows. Positive frequency modes with respect to global time ϕ^E are analytic in the lower half plane when expressed in terms of Kruskal coordinates. These modes are decomposed into modes with positive and negative frequencies with respect to time in the south and north static patches [20]. The decomposition takes the following form [10]

$$\phi_\omega^E = \phi_\omega^S + e^{-\pi\omega}\phi_\omega^N \ , \tag{14}$$

i.e., the positive-frequency component ϕ_ω^S comes from the southern causal diamond, while the negative-frequency component ϕ_ω^N comes from the northern causal diamond, where t runs backward. There is a second linear combination that has positive frequency with respect to global time,

$$\tilde{\phi}_\omega^E = \phi_\omega^{N*} + e^{-\pi\omega}\phi_\omega^{S*} \ , \tag{15}$$

where the roles of north and south are swapped.

Now both ϕ_ω^E and $\tilde{\phi}_\omega^E$ annihilate the Euclidean vacuum when $\omega > 0$. It is then possible to show the Euclidean vacuum can be written as

$$
\begin{aligned}
|E\rangle &= \prod_{\omega>0}(1 - e^{-2\pi\omega})^{1/2}\exp\left(e^{-\pi\omega}a_\omega^{S\dagger}a_\omega^N\right)|S\rangle \otimes |N\rangle \\
&= \prod_{\omega>0}(1 - e^{-2\pi\omega})^{1/2}\sum_{n_\omega=0}^{\infty}e^{-\pi\omega n_\omega}|n_\omega, S\rangle \otimes |n_\omega, N\rangle
\end{aligned}
$$

where $|N\rangle$ is the northern vacuum annihilated by ϕ_ω^{N*} and $|S\rangle$ is the southern vacuum annihilated by ϕ_ω^S. The state $|n_\omega, S\rangle$ denotes occupation number n_ω in the ω mode with respect to the southern vacuum state, and likewise for $|n_\omega, N\rangle$. It is then straightforward to trace over modes in the northern diamond, to obtain a density matrix for modes within only the southern causal diamond

$$\rho_S = \prod_{\omega>0}(1 - e^{-2\pi\omega})\sum_{n_\omega=0}^{\infty}e^{-2\pi\omega n_\omega}|n_\omega, S\rangle \otimes \langle n_\omega, S| \ . \tag{16}$$

This takes the form of a thermal density matrix, where the units are such that the temperature of de Sitter is $1/2\pi$. The free energy associated with this entanglement is

$$e^{-2\pi F} = \text{Tr } e^{-2\pi\omega n_\omega} \ . \tag{17}$$

Using the unitary Virasoro representations we have constructed, we now can map each step in the above entropy calculation into a counting of operators in the boundary CFT with a particular weighting. For example, the conjugate of each of the modes (14) and (15) is dual to a CFT operator $\mathcal{O}_\omega^E$ or $\tilde{\mathcal{O}}_\omega^E$ which is a linear combination of the operators dual to single particle modes described above. These operators create modes with correlations between the north and south static patches. Note that since the bulk theory is formulated around the Euclidean vacuum, it is not possible to avoid this correlation and create modes localized purely in one static patch. A similar mapping will exist for all the multiparticle states. From the CFT viewpoint, the partition function (17) corresponds to the following weighted sum over the operator content of the CFT

$$e^{-2\pi F} = \text{Tr } e^{-\pi i(L_{-1}-L_1)} \ ,$$

where the trace is over the full set of operators dual to the scalar field Fock space, as described in table I. The trace is defined with the restriction that the eigenvalue of $i(L_{-1} - L_1)$ is positive, corresponding to the $\omega > 0$ condition in (16). This at least solves one aspect of the entropy counting problem in dS/CFT.

The problem is that the resulting entropy is divergent, due to a continuum of states with finite $i(L_{-1} - L_1)$ localized near the cosmological horizon. These may be cutoff with analogs of the brick-wall cutoff of 't Hooft [21]. Such calculations have been performed in [22]. All these may now be interpreted in terms of the boundary CFT using the dictionary we have provided – but we need new physical input to really understand the role of the cutoff. We hope to explore these issues in the future, applying ideas of q-deformation [4, 5, 23, 24].

Appendix A: CENTRAL CHARGE

For $U = ie^{in\theta}$, we obtain

$$[: L_U :, : L_{U^*} :] = \cosh t \int d\theta \, d\theta' \, e^{in(\theta'-\theta)} i\delta'(\theta - \theta') \left(\dot{\phi}(\theta)\phi'(\theta') + \dot{\phi}(\theta')\phi'(\theta) \right) . \tag{A1}$$

To make the normal ordering terms on the right hand side of (A1) well-defined, one may use a
$i\epsilon$ prescription. It is also necessary to regulate the $\delta'(\theta-\theta')$ factor using the same prescription.
A potential central term in the algebra then arises from $\left\langle \dot{\phi}(\theta)\phi'(\theta') + \dot{\phi}(\theta')\phi'(\theta) \right\rangle$ in the limit
that points coincide. In this limit, we can evaluate using

$$\langle \phi(\theta, t)\phi(\theta', t') \rangle = -\frac{1}{4\pi} \log((t - t' - i\epsilon)^2 - \cosh t \cosh t'(\theta - \theta')^2) .$$

Direct computation for the correlator of interest shows at equal times the $i\epsilon$ terms cancel
and

$$\left\langle \dot{\phi}(\theta)\phi'(\theta') + \dot{\phi}(\theta')\phi'(\theta) \right\rangle = 0 .$$

Therefore the central term vanishes.

Appendix B: VIRASORO GENERATORS

In this appendix we compute the expansion of the Virasoro generators in terms of cre-
ation and annihilation operators, and demonstrate the time-independence of the coefficients.
Consider first the case of dS_2 in spherical coordinates, where the mode expansion is given by
(1) with $l \to k \in \mathbb{Z}$ and $Y_k(\theta) = e^{ik\theta}/\sqrt{2\pi}$. Using the asymptotic form (2) and substituting
into (10) with $U = ie^{in\theta}$, the Virasoro generator $: L_n :$ is expressed as a sum over k of an
expression quadratic in creation/annihilation operators (which by definition is meant to be
normal-ordered). At first sight the coefficients in this sum appear to have some remaining
time-dependence, but this dependence cancels when one combines the two distinct terms in

the sum that contribute to a given aa, $a^\dagger a$, or $a^\dagger a^\dagger$ product, and makes use of the identity $h_\pm - h_\pm^2 - m^2 = 0$. The end result is

$$
\begin{aligned}
2i : L_n : \; = \; & \sum_{k \geq [(n+1)/2]} a_k^\dagger a_{-k+n}^\dagger \left[\left(n(h_- - 2m^2) + 2i\mu k \right) A_{-k+n}^* B_k^* + \left(n(h_+ - 2m^2) - 2i\mu k \right) A_k^* B_{-k+n}^* \right] \\
& + \sum_{k \geq [(n+1)/2]} a_{-k} a_{k-n} \left[\left(n(h_+ - 2m^2) - 2i\mu k \right) A_{k-n} B_{-k} + \left(n(h_- - 2m^2) + 2i\mu k \right) A_{-k} B_{k-n} \right] \\
& + \sum_{k=-\infty}^{\infty} a_{n-k}^\dagger a_{-k} \left[\left(n(h_- - 2m^2) + 2i\mu k \right) A_{n-k}^* A_{-k} + \left(n(h_+ - 2m^2) - 2i\mu k \right) B_{n-k}^* B_{-k} \right]
\end{aligned}
$$

If n is even, the first term of the sums in the first and second lines corresponds to $k = n/2$, and should actually have a coefficient that is only half as big as the other terms.

For dS_2 in planar coordinates (where the modes are given by (6) and (7)) one finds a completely analogous result, with the integer n that labels the Virasoro generator replaced by a real number p, the sums over k replaced by integrals $\int_{(p+1)/2}^{\infty} dk/2\pi$ and $\int_{-\infty}^{\infty} dk/2\pi$, and the factor of 2 removed from the left-hand side of the equation.

It should be noted the coefficients of the $a^\dagger a^\dagger$ and the aa terms vanish for the global generators $L_{0,\pm 1}$ for dS_2 in spherical coordinates. The same is true for the global generators in flat coordinates, but these are nontrivial linear combinations in the basis of Virasoro generators defined above, corresponding to $U = -1, -z, -z^2$. The global generators therefore preserve particle number, and one sees the one-particle sector of the Fock space transforms as a unitary principal series representation, as expected.

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[28] Related work by Neretin extends the complementary series representations of $SL(2, \mathbb{R})$ to the Virasoro algebra with vanishing central charge [25, 26, 27].

[29] If the dual CFT were to be radially quantized, as is standard, e.g., in string theory applications, this contour-independence would of course identify the H's as conserved charges. We would like to emphasize, however, that in the dS_d/CFT_{d-1} context there is a priori no time direction in the CFT, and so spatial integrals are naturally $(d-1)$- rather than $(d-2)$-dimensional.